PG – 362

II Semester M.Sc. Degree Examination, June 2015
(CBCS)
MATHEMATICS
M 201 T : Algebra – II

Time : 3 Hours Max. Marks : 70

Instructions : *1) Answer **any 5** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define the Jacobson radical J(A) of a commutative ring A.
Prove that $x \in J(A)$ if and only if $1 - xy$ is a unit in A for all $x \in A$.

b) Let $I_1, I_2, \ldots, I_n$ be the ideals of a ring A which are pairwise coprime, that is I_j and I_k are coprime for $j \neq k$, then show that $\prod_{k=1}^{n} I_k = \bigcap_{k=1}^{n} I_k$.

c) Prove that spec A is compact. **(5+5+4)**

2. a) If $L \supseteq M \supseteq B$ are A-modules, then show that $\left(L/N\right)/\left(M/N\right) \cong L/M$.

b) Show that M is a finitely generated A-module if and only if M is isomorphic to a quotient of a finitely generated free A-module.

c) State and prove Nakayama Lemma. **(4+4+6)**

3. a) Define a simple module.
Show that an A-module M is simple if and only if $M \cong A/I$ for some maximal ideal I of A.

b) Prove that the sequence :
$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$ is exact if and only if, for all A-modules N, the sequence $0 \longrightarrow \text{Hom}(M'', N) \xrightarrow{\bar{v}} \text{Hom}(M, N) \xrightarrow{\bar{u}} \text{Hom}(M', N)$ is exact. **(7+7)**

4. a) Show that M is a Noetherian A-module if and only if every sub module of M is finitely generated.
   b) Show that an Artinian integral domain is a field.
   c) State and prove Hilbert basis theorem. **(5+3+6)**

5. a) If L is a finite extension of K and K is a finite extension of F, then show that L is a finite extension of F.
   b) Prove that the elements in an extension K of a field F which are algebraic over F form a subfield of K.
   c) Prove that $Q\left(\sqrt{2}, \sqrt{3}\right) = Q\left(\sqrt{2} + \sqrt{3}\right)$. **(6+4+4)**

6. a) Let f(x) $\in$ F[x] be a polynomial of degree $n \geq 1$. Then prove that there is an extension E of F of degree atmost n! in which f(x) has n-roots.
   b) Determine the spliting field of $x^3 - 2$ over the field Q.
   c) Prove that it is impossible to trisect 60° by using straight edge and compass. **(5+4+5)**

7. a) Show that the polynomial f(x) $\in$ F[x] has multiple roots if and only if f(x) and $f'(x)$ have a non-trivial common factors.
   b) Prove that any finite extension of a field F of characteristic 0 is a simple extension.
   c) Show that any field of characteristic zero is perfect field. **(6+4+4)**

8. a) Define a fixed field.
   Let G be a subgroup of the group of all automorphisms of a field K. Then show that fixed field of G is a subfield of K.
   b) State and prove the fundamental theorem of Galois theory. **(4+10)**